MEIXNER POLYNOMIALS IN SEVERAL VARIABLES
SATISFYING BISPECTRAL DIFFERENCE EQUATIONS

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ABSTRACT. We construct a set $\mathfrak{M}_d$ whose points parametrize families of Meixner polynomials in d variables. There is a natural bispectral involution $\mathfrak{b}$ on $\mathfrak{M}_d$ which corresponds to a symmetry between the variables and the degree indices of the polynomials. We define two sets of d commuting partial difference operators diagonalized by the polynomials. One of the sets consists of difference operators acting on the variables of the polynomials and the other one on their degree indices, thus proving their bispectrality. The two sets of partial difference operators are naturally connected via the involution $\mathfrak{b}$.

1. INTRODUCTION

One of the reasons why the classical orthogonal polynomials $p_n(x)$ of a discrete variable x appear in numerous applications is the fact they are eigenfunctions of a second-order difference operator acting on x. A detailed account of their characteristic properties and application can be found in [16].

The last few years there have been advances in different aspects of the classification and the construction of orthogonal polynomials in several variables which are eigenfunctions of partial difference operators, see for instance [4, 6, 9, 10, 11, 12]. The difficulty in the multivariate case is that the polynomials are no longer uniquely determined by the orthogonality measure (up to a multiplicative constant), and therefore their spectral properties depend on the way we apply the Gram-Schmidt process, see [3] for the general theory.

Following the pioneering work of Duistermaat and Grünbaum [2], we say that a family $\{P_{n_1,\ldots,n_d}(x_1,\ldots,x_d)\}$ of polynomials in d variables $x_1,\ldots,x_d$ with degree indices $n_1,\ldots,n_d$ solves a discrete-discrete bispectral problem, if there exist commuting partial difference operators $\mathcal{L}_1^x,\ldots,\mathcal{L}_d^x$ acting on the variables $x_1,\ldots,x_d$ with coefficients independent of $n_1,\ldots,n_d$ and commuting partial difference operators $\mathcal{L}_1^n,\ldots,\mathcal{L}_d^n$ acting on the degree indices $n_1,\ldots,n_d$ with coefficients independent of $x_1,\ldots,x_d$ such that

$$\mathcal{L}_j^x P_{n_1,\ldots,n_d}(x_1,\ldots,x_d) = \lambda_j(n_1,\ldots,n_d)P_{n_1,\ldots,n_d}(x_1,\ldots,x_d) \qquad (1.1a)$$

$$\mathcal{L}_j^n P_{n_1,\ldots,n_d}(x_1,\ldots,x_d) = \mu_j(x_1,\ldots,x_d)P_{n_1,\ldots,n_d}(x_1,\ldots,x_d), \qquad (1.1b)$$

for $j = 1, 2, \ldots, d$, where $\lambda_j(n_1,\ldots,n_d)$ are independent of $x_1,\ldots,x_d$ and $\mu_j(x_1,\ldots,x_d)$ are independent of $n_1,\ldots,n_d$. Within the context of orthogonal polynomials one is often interested in the commuting operators corresponding to multiplications by the independent variables x_j and therefore $\mu_j(x_1,\ldots,x_d) = x_j$ is one natural choice for the eigenvalues in (1.1b). On the other hand, even in the one-variable case, the

eigenvalues in (1.1a) can be quadratic functions (e.g. in the case of Hahn polynomials) and in higher dimensions, they can depend on some or all of the chosen degree indices $n_1, \ldots, n_d$, see for instance the families discussed in [4, 9].

Recently [10] we proved that all families of multivariate Krawtchouk polynomials defined by Griffiths [5] are bispectral. We used a Lie-theoretic approach which naturally led to the construction of the partial difference operators in equations (1.1). Moreover, we showed how other known families of bispectral multivariate Krawtchouk polynomials fit within the above theory. The construction was a natural extension of an earlier joint work with Terwilliger [11] for the bivariate polynomials defined in [8]. We note that different proofs of the orthogonality properties of the multivariate Krawtchouk polynomials as well as interesting probabilistic applications and connections to Gelfand pairs and character algebras were discussed in the papers [7, 13, 14, 15].

In the present paper we give a similar parametrization of bispectral multivariate Meixner polynomials and the corresponding partial difference operators. The main result concerning the bispectrality of the multivariate Meixner polynomials constructed here is analogous to one in the Krawtchouk case [10], but the proof is different.

The paper is organized as follows. In the next section, we define a set $\mathfrak{M}_d$ and for every $\mathfrak{m} \in \mathfrak{M}_d$ and $\beta \in \mathbb{C} \setminus (-\mathbb{N}_0)$ we construct multivariate Meixner polynomials $\{P_{n_1,\ldots,n_d}(x_1, \ldots, x_d; \mathfrak{m}, \beta) : n_j \in \mathbb{N}_0\}$, mutually orthogonal on $\mathbb{N}_0^d$ with respect to a weight function of the form

$$W(x) = (\beta)_{\sum_{j=1}^d x_j} \prod_{j=1}^d \frac{c_j^{x_j}}{x_j!}.$$

The construction is based on a generating function and is similar to one used by Griffiths [5] in the case of the Krawtchouk polynomials. In Section 3 we give a hypergeometric representation for the multivariate Meixner polynomials which yields the duality between the variables and the degree indices of the polynomials. The key ingredient is to "expand" the generating function appropriately and to regroup the terms by changing the summation indices. The computation is essentially the same as the one in the Krawtchouk case which can found in the work of Mizukawa and Tanaka [15], except that the expansion of the generating series now is infinite. In Section 4 we construct the commuting partial difference operators $\{\mathcal{L}_j^x\}_{j=1}^d$, $\{\mathcal{L}_j^n\}_{j=1}^d$ and we prove that they are diagonalized by the polynomials $\{P_{n_1,\ldots,n_d}(x_1, \ldots, x_d; \mathfrak{m}, \beta) : n_j \in \mathbb{N}_0\}$ by using the generating function and the duality. Finally, we list two explicit families of Meixner polynomials in arbitrary dimension in Section 5. In the first example, we explain how we need to pick the point $\mathfrak{m} \in \mathfrak{M}_d$ in order to obtain the multivariate Meixner polynomials defined in [12]. In the second example, we construct a point $\mathfrak{m} \in \mathfrak{M}_d$ which depends on one free parameter. The corresponding Meixner polynomials can be thought of as analogs of the the multivariate Krawtchouk polynomials used in [1].

2. MULTIVARIATE MEIXNER POLYNOMIALS

2.1. **Basic notations and definitions.** Throughout the paper we shall use standard multi-index notations. For instance, if $x = (x_1, x_2, \ldots, x_d) \in \mathbb{C}^d$ and $n =$

$(n_1, n_2, \ldots, n_d) \in \mathbb{N}_0^d$ then

$$x^n = x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d}, \qquad n! = n_1! n_2! \cdots n_d!,$$

and $|x| = x_1 + x_2 + \cdots + x_d$.

First we introduce the set $\mathfrak{M}_d$ whose points parametrize families of multivariate Meixner polynomials. In most applications, the parameters are real, but since the partial difference equations studied here have natural extensions for complex numbers, we work below with $\mathbb{C}$ rather than $\mathbb{R}$.

Definition 2.1. Let $\mathfrak{M}_d$ denote the set of 4-tuples $(c_0, C, \tilde{C}, U)$, where c_0 is a nonzero complex number and $C, \tilde{C}, U$ are $(d+1) \times (d+1)$ matrices with complex entries satisfying the following conditions:

(i) $C = \mathrm{diag}(1, -c_1, -c_2, \ldots, -c_d)$ and $\tilde{C} = \mathrm{diag}(1, -\tilde{c}_1, -\tilde{c}_2 \ldots, -\tilde{c}_d)$ are diagonal;

(ii) $U = (u_{i,j})_{0 \leq i,j \leq d}$ is such that $u_{0,j} = u_{j,0} = 1$ for all $j = 0, 1, \ldots, d$, i.e.

$$U = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & u_{1,1} & u_{1,2} & \ldots & u_{1,d} \\ \vdots & & & & \\ 1 & u_{d,1} & u_{d,2} & \ldots & u_{d,d} \end{pmatrix}; \tag{2.1}$$

(iii) The following matrix equation holds

$$U^t C U \tilde{C} = c_0 I_{d+1}, \tag{2.2}$$

where I_{d+1} denotes the identity $(d+1) \times (d+1)$ matrix.

Note that c_0 is a nonzero number and therefore (2.2) implies that the matrices C, $\tilde{C}$ and U in Definition 2.1 are invertible. We shall denote by c and $\tilde{c}$ the d-dimensional vectors

$$c = (c_1, c_2, \ldots, c_d), \qquad \tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_d).$$

Comparing the (i, j) entries on both sides of equation (2.2) for $j = 0$ and $j > 0$ we obtain the following identities:

$$\sum_{k=1}^d c_k u_{k,i} = 1 - c_0 \delta_{i,0} \tag{2.3a}$$

$$\sum_{k=1}^d c_k u_{k,i} u_{k,j} = 1 + \frac{c_0}{\tilde{c}_j} \delta_{i,j}. \tag{2.3b}$$

Remark 2.2. We shall use later the involution $\mathfrak{b}$ on $\mathfrak{M}_d$ defined by

$$\mathfrak{b} : \mathfrak{m} = (c_0, C, \tilde{C}, U) \to \tilde{\mathfrak{m}} = (c_0, \tilde{C}, C, U^t). \tag{2.4}$$

In particular, using (2.3a) with $i = 0$ and $\mathfrak{b}$ we see that

$$c_0 = 1 - |c| = 1 - |\tilde{c}|. \tag{2.5}$$

Remark 2.3. If we start with nonzero numbers $c_1, c_2, \ldots, c_d$ such that $c_0 = 1 - |c| \neq 0$ we can define a point $\mathfrak{m} = (c_0, C, \tilde{C}, U) \in \mathfrak{M}_d$ as follows. First, we construct a basis $v_0, v_1, \ldots, v_d$ of column vectors in $\mathbb{C}^{d+1}$ mutually orthogonal with respect to the bilinear form $\langle w_1, w_2 \rangle = w_1^t C w_2$ such that $v_0 = (1, 1, \ldots, 1)^t$, the 0-th coordinate of v_j is 1 and $\langle v_j, v_j \rangle \neq 0$ for $j = 1, 2, \ldots, d$. Then the matrix U with columns $v_0, v_1, \ldots, v_d$ is of the form given in (2.1) and the diagonal matrix $\tilde{C}$ is

uniquely determined by (2.2). If $d = 1$, then $v_1 = (1, 1/c_1)^t$ is uniquely determined from c_1. However, when $d > 1$ we have $d(d-1)/2$ degrees of freedom in choosing U.

In the rest of the paper we use $\{z_j\}_{j=1}^d$ and $\{w_j\}_{j=1}^d$ to denote formal complex variables which are sufficiently close to 0. In all exponents below we fix the principal branch of the logarithmic function.

For every $\mathfrak{m} \in \mathfrak{M}_d$ and $\beta \in \mathbb{C}$ we consider the function of $x = (x_1, \ldots, x_d)$, $z = (z_1, \ldots, z_d)$

$$G(x, z; \mathfrak{m}, \beta) = (1 - |z|)^{-\beta - |x|} \prod_{i=1}^d \left(1 - \sum_{j=1}^d u_{i,j} z_j \right)^{x_i}. \tag{2.6}$$

Definition 2.4. For $\mathfrak{m} \in \mathfrak{M}_d$ and $\beta \in \mathbb{C} \setminus (-\mathbb{N}_0)$ we define multivariate Meixner polynomials $\{P_n(x; \mathfrak{m}, \beta) : n \in \mathbb{N}_0^d\}$ by expanding $G(x, z; \mathfrak{m}, \beta)$ in a neighborhood of $z = 0$ as follows

$$G(x, z; \mathfrak{m}, \beta) = \sum_{n \in \mathbb{N}_0^d} \frac{(\beta)_{|n|}}{n!} P_n(x; \mathfrak{m}, \beta) z^n. \tag{2.7}$$

2.2. **Orthogonality relations.** We show next that the polynomials $\{P_n(x; \mathfrak{m}, \beta) : n \in \mathbb{N}_0^d\}$ defined above are mutually orthogonal with respect to the weight $(\beta)_{|x|} c^x / x!$ on $\mathbb{N}_0^d$. More precisely, using the notations introduced so far, the following theorem holds.

Theorem 2.5. *Suppose that $|c_1| + |c_2| + \cdots + |c_d| < 1$. Then for $n, m \in \mathbb{N}_0^d$ we have*

$$\sum_{x \in \mathbb{N}_0^d} P_n(x; \mathfrak{m}, \beta) P_m(x; \mathfrak{m}, \beta) \frac{(\beta)_{|x|}}{x!} c^x = \frac{c_0^{-\beta} n!}{(\beta)_{|n|} \tilde{c}^n} \delta_{n,m}. \tag{2.8}$$

Proof. Using the formula

$$(1 - |z|)^{-\gamma} = \sum_{k \in \mathbb{N}_0^d} \frac{(\gamma)_{|k|}}{k!} z^k, \tag{2.9}$$

we find

$$\sum_{x \in \mathbb{N}_0^d} G(x, z; \mathfrak{m}, \beta) G(x, w; \mathfrak{m}, \beta) \frac{(\beta)_{|x|}}{x!} c^x$$

$$= \left[(1 - |z|)(1 - |w|) - \sum_{i=1}^d c_i \left(1 - \sum_{j=1}^d u_{i,j} z_j \right) \left(1 - \sum_{s=1}^d u_{i,s} w_s \right) \right]^{-\beta}. \tag{2.10}$$

If we set $Z = (-1, z_1, \ldots, z_d)^t$ and $W = (-1, w_1, \ldots, w_d)^t$, then the expression in the big parentheses on the right-hand side above is equal to $(UW)^t C(UZ)$. Using (2.2) we find

$$(UW)^t C(UZ) = c_0 W^t \tilde{C}^{-1} Z = c_0 \left[1 - \sum_{j=1}^d \frac{z_j w_j}{\tilde{c}_j} \right].$$

Thus, (2.10) gives

$$\sum_{x \in \mathbb{N}_0^d} G(x, z; \mathfrak{m}, \beta) G(x, w; \mathfrak{m}, \beta) \frac{(\beta)_{|x|}}{x!} c^x = c_0^{-\beta} \left[1 - \sum_{j=1}^d \frac{z_j w_j}{\tilde{c}_j} \right]^{-\beta}.$$

The proof now follows by expanding the right side of the last equation using (2.9) and comparing the coefficients of $z^n w^m$ on both sides of the resulting identity. $\square$

3. Hypergeometric representation

We denote by $\mathbb{M}_d$ the set of all $d \times d$ matrices with entries in $\mathbb{N}_0$.

Theorem 3.1. *We have*

$$P_n(x; \mathfrak{m}, \beta) = \sum_{A=(a_{i,j}) \in \mathbb{M}_d} \frac{\prod_{j=1}^d (-n_j)_{\sum_{i=1}^d a_{i,j}} \prod_{i=1}^d (-x_i)_{\sum_{j=1}^d a_{i,j}}}{(\beta)_{\sum_{i,j=1}^d a_{i,j}}} \prod_{i,j=1}^d \frac{(1-u_{i,j})^{a_{i,j}}}{a_{i,j}!}.$$
(3.1)

Note that $(-n_j)_{\sum_{i=1}^d a_{i,j}} = 0$ when $\sum_{i=1}^d a_{i,j} > n_j$. Therefore the sum in (3.1) is finite, using only matrices A with nonnegative integer entries satisfying $\sum_{i=1}^d a_{i,j} \leq n_j$ for $j = 1, \ldots, d$.

Proof of Theorem 3.1. For every $i \in \{1, 2, \ldots, d\}$ we have

$$1 - \sum_{j=1}^d u_{i,j} z_j = (1 - |z|) \left(1 + \sum_{j=1}^d \frac{(1-u_{i,j}) z_j}{1 - |z|} \right).$$

Taking the x_i-th power of the above equation and expanding the second term using (2.9) we obtain

$$\left(1 - \sum_{j=1}^d u_{i,j} z_j \right)^{x_i} = (1 - |z|)^{x_i} \sum_{k_i \in \mathbb{N}_0^d} \frac{(-1)^{|k_i|} (-x_i)_{|k_i|}}{k_i!} \frac{\eta_i^{k_i} z^{k_i}}{(1 - |z|)^{|k_i|}},$$

where we have set $\eta_i = (1 - u_{i,1}, 1 - u_{i,2}, \ldots, 1 - u_{i,d})$. Substituting the last formula into the right-hand side of (2.6) we obtain

$$G(x, z; \mathfrak{m}, \beta) = \sum_{k_1, k_2, \ldots, k_d \in \mathbb{N}_0^d} (1 - |z|)^{-\beta - \sum_{i=1}^d |k_i|} (-1)^{\sum_{i=1}^d |k_i|}$$

$$\times \left(\prod_{i=1}^d \frac{(-x_i)_{|k_i|}}{k_i!} \eta_i^{k_i} \right) z^{\sum_{i=1}^d k_i}.$$

Expanding also $(1 - |z|)^{-\beta - \sum_{i=1}^d |k_i|}$ we get

$$G(x, z; \mathfrak{m}, \beta) = \sum_{l, k_1, k_2, \ldots, k_d \in \mathbb{N}_0^d} \frac{(-1)^{\sum_{i=1}^d |k_i|} \left(\beta + \sum_{i=1}^d |k_i| \right)_{|l|}}{l!}$$

$$\times \left(\prod_{i=1}^d \frac{(-x_i)_{|k_i|}}{k_i!} \eta_i^{k_i} \right) z^{l + \sum_{i=1}^d k_i}.$$
(3.2)

Let us denote by $a_{i,j}$ the entries of the vector k_i, and by l_j the entries of l, i.e. $k_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,d})$ and $l = (l_1, l_2, \ldots, l_d)$. Then

$$\frac{(-1)^{\sum_{i=1}^d |k_i|} \left(\beta + \sum_{i=1}^d |k_i|\right)_{|l|}}{l!} = \frac{(\beta)_{|l|+\sum_{i,j=1}^d a_{i,j}}}{(\beta)_{\sum_{i,j=1}^d a_{i,j}}} \prod_{j=1}^d \frac{\left(-l_j - \sum_{i=1}^d a_{i,j}\right)_{\sum_{i=1}^d a_{i,j}}}{\left(l_j + \sum_{i=1}^d a_{i,j}\right)!}.$$
(3.3)

The proof now follows by plugging (3.3) in the first line of (3.2) and by replacing the sum over l with a sum over $n = (n_1, n_2, \ldots, n_d)$ where $n_j = l_j + \sum_{i=1}^d a_{i,j}$. $\square$

As an immediate corollary of the hypergeometric representation (3.1) and the involution $\mathfrak{b}$ on $\mathfrak{M}_d$ in Remark 2.2 we see that the polynomials $P_n(x; \mathfrak{m}, \beta)$ possess a duality between the variables $\{x_j\}$ and the indices $\{n_j\}$.

Corollary 3.2. *For $n, x \in \mathbb{N}_0^d$ we have*

$$P_n(x; \mathfrak{m}, \beta) = P_x(n, \tilde{\mathfrak{m}}, \beta).$$
(3.4)

4. BISPECTRALITY

Let $\{e_1, e_2, \ldots, e_d\}$ be the standard basis for $\mathbb{C}^d$. For every $i \in \{1, 2, \ldots, d\}$ we denote by E_{x_i} and E_{n_i} the customary shift operators acting on functions of $x = (x_1, x_2, \ldots, x_d)$ and $n = (n_1, n_2, \ldots, n_d)$, respectively, as follows

$$E_{x_i} f(x) = f(x + e_i) \quad \text{and} \quad E_{n_i} g(n) = g(n + e_i).$$

For fixed $\mathfrak{m} \in \mathfrak{M}_d$, $\beta \in \mathbb{C}$ and $i \in \{1, 2, \ldots, d\}$ we define the following difference operators

$$\mathcal{L}_i^x = \frac{\tilde{c}_i}{c_0} \sum_{1 \le k \ne l \le d} c_k u_{k,i} u_{l,i} x_l \left(E_{x_k} E_{x_l}^{-1} - \mathrm{Id}\right)$$

$$- \frac{\tilde{c}_i}{c_0} \sum_{l=1}^d u_{l,i} x_l \left(E_{x_l}^{-1} - \mathrm{Id}\right) - \frac{\tilde{c}_i}{c_0} \sum_{k=1}^d c_k u_{k,i} (\beta + |x|) \left(E_{x_k} - \mathrm{Id}\right), \qquad (4.1a)$$

and

$$\mathcal{L}_i^n = \frac{c_i}{c_0} \sum_{1 \le k \ne l \le d} \tilde{c}_k u_{i,k} u_{i,l} n_l \left(E_{n_k} E_{n_l}^{-1} - \mathrm{Id}\right)$$

$$- \frac{c_i}{c_0} \sum_{l=1}^d u_{i,l} n_l \left(E_{n_l}^{-1} - \mathrm{Id}\right) - \frac{c_i}{c_0} \sum_{k=1}^d \tilde{c}_k u_{i,k} (\beta + |n|)(E_{n_k} - \mathrm{Id}), \qquad (4.1b)$$

where Id denotes the identity operator. We show below that these operators are diagonalized by the polynomials $P_n(x; \mathfrak{m}, \beta)$, thus providing solutions to the bispectral problem. Note that the backward shift operators $E_{n_l}^{-1}$ in (4.1b) are multiplied by n_l, and therefore these expressions will vanish when $n_l = 0$. Thus $\mathcal{L}_i^n$ is a well-defined operator acting on functions of $n \in \mathbb{N}_0^d$. Similarly, in view of the orthogonality (2.8), it is natural to consider the polynomials for $x \in \mathbb{N}_0^d$ and the operators $\mathcal{L}_i^x$ will involve evaluations of $P_n(x; \mathfrak{m}, \beta)$ only at points x with nonnegative integer coordinates.

Theorem 4.1. *For every $i \in \{1, 2, \ldots, d\}$ the following spectral equations hold*

$$\mathcal{L}_i^x P_n(x; \mathfrak{m}, \beta) = n_i P_n(x; \mathfrak{m}, \beta), \tag{4.2a}$$

$$\mathcal{L}_i^n P_n(x; \mathfrak{m}, \beta) = x_i P_n(x; \mathfrak{m}, \beta). \tag{4.2b}$$

Proof. Fix $i \in \{1, 2, \ldots, d\}$ and let us denote

$$\mathcal{L}_{i,l}^x = \mathrm{Id} - E_{x_l}^{-1} + \sum_{k \in \{1,\ldots,d\}\setminus\{l\}} c_k u_{k,i} \left(E_{x_k} E_{x_l}^{-1} - \mathrm{Id} \right) \text{ for every } l = 1, 2, \ldots, d$$

and

$$\mathcal{L}_{i,+}^x = -\frac{\tilde{c}_i}{c_0} \sum_{k=1}^d c_k u_{k,i} (\beta + |x|) \left(E_{x_k} - \mathrm{Id} \right).$$

Then

$$\mathcal{L}_i^x = \frac{\tilde{c}_i}{c_0} \sum_{l=1}^d u_{l,i} x_l \mathcal{L}_{i,l}^x + \mathcal{L}_{i,+}^x. \tag{4.3}$$

Using the definition (2.6) of $G(x, z; \mathfrak{m}, \beta)$ we find

$$\frac{1}{G(x, z; \mathfrak{m}, \beta)} \sum_{k \in \{1,\ldots,d\}\setminus\{l\}} c_k u_{k,i} \left(E_{x_k} E_{x_l}^{-1} - \mathrm{Id} \right) G(x, z; \mathfrak{m}, \beta)$$

$$= \frac{1}{\left(1 - \sum_{j=1}^d u_{l,j} z_j\right)} \sum_{k \in \{1,\ldots,d\}\setminus\{l\}} \sum_{j=1}^d (c_k u_{k,i} u_{l,j} - c_k u_{k,i} u_{k,j}) z_j$$

(interchanging the sums)

$$= \frac{1}{\left(1 - \sum_{j=1}^d u_{l,j} z_j\right)} \sum_{j=1}^d \sum_{k=1}^d (c_k u_{k,i} u_{l,j} - c_k u_{k,i} u_{k,j}) z_j$$

(using equations (2.3))

$$= \frac{1}{\left(1 - \sum_{j=1}^d u_{l,j} z_j\right)} \sum_{j=1}^d \left(u_{l,j} - \frac{c_0}{\tilde{c}_j} \delta_{i,j} - 1 \right) z_j$$

$$= \frac{1}{\left(1 - \sum_{j=1}^d u_{l,j} z_j\right)} \left(\sum_{j=1}^d (u_{l,j} - 1) z_j - \frac{c_0}{\tilde{c}_i} z_i \right).$$

From the last relation, the definition of $\mathcal{L}_{i,l}^x$ and

$$\frac{1}{G(x, z; \mathfrak{m}, \beta)} (\mathrm{Id} - E_{x_l}^{-1}) G(x, z; \mathfrak{m}, \beta) = \frac{\sum_{j=1}^d (1 - u_{l,j}) z_j}{1 - \sum_{j=1}^d u_{l,j} z_j}$$

it follows that

$$\frac{1}{G(x, z; \mathfrak{m}, \beta)} \mathcal{L}_{i,l}^x G(x, z; \mathfrak{m}, \beta) = -\frac{c_0 z_i}{\tilde{c}_i \left(1 - \sum_{j=1}^d u_{l,j} z_j\right)}. \tag{4.4}$$

A similar computation shows that

$$\frac{1}{G(x, z; \mathfrak{m}, \beta)} \mathcal{L}_{i,+}^x G(x, z; \mathfrak{m}, \beta) = \frac{(\beta + |x|) z_i}{1 - |z|}. \tag{4.5}$$

Using equations (4.3), (4.4) and (4.5) we see that

$$\mathcal{L}_i^x G(x, z; \mathfrak{m}, \beta) = z_i \frac{\partial}{\partial z_i} G(x, z; \mathfrak{m}, \beta), \qquad (4.6)$$

which combined with (2.7) completes the proof of (4.2a). The proof of (4.2b) follows from the duality established in Corollary 3.2. Indeed, if $x \in \mathbb{N}_0^d$, then $P_n(x; \mathfrak{m}, \beta) = P_x(n, \tilde{\mathfrak{m}}, \beta)$ and equation (4.2b) follows from (4.2a). Moreover, for fixed $n \in \mathbb{N}_0^d$ both sides of (4.2b) are polynomials in x of total degree at most $|n|+1$, hence if the equality holds for all $x \in \mathbb{N}_0^d$ it will be true for arbitrary $x \in \mathbb{C}^d$. $\qquad\square$

5. Some examples

In this section we illustrate how specific points in $\mathfrak{M}_d$ lead to families of multivariate Meixner polynomials or analogs of multivariate Krawtchouk polynomials which have already appeared in the literature in different applications.

5.1. Let us fix nonzero complex numbers $c_1, \ldots, c_d$ and let $c_0 = 1 - |c|$. We define $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_d$ in terms of $c_1, \ldots, c_d$ as follows:

$$\tilde{c}_k = \frac{c_k c_0}{(1 - \sum_{j=k+1}^d c_j)(1 - \sum_{j=k}^d c_j)} \text{ for } k = 1, 2, \ldots, d. \qquad (5.1)$$

We shall assume that $\{c_j\}$ are such that the denominators in (5.1) do not vanish. Next we define a $(d+1) \times (d+1)$ matrix $U = (u_{i,j})$ with entries

$$u_{i,j} = \delta_{0,i}, \qquad\qquad\qquad \text{when } i < j, \qquad (5.2a)$$
$$u_{i,j} = 1, \qquad\qquad\qquad \text{when } i > j, \qquad (5.2b)$$
$$u_{0,0} = 1, \qquad\qquad\qquad\qquad (5.2c)$$
$$u_{i,i} = -\frac{1 - \sum_{k=i+1}^d c_k}{c_i}, \qquad \text{for } i = 1, 2, \ldots, d. \qquad (5.2d)$$

Thus, U is a matrix of the form (2.1) where the remaining entries are 0's and 1's above and below the diagonal respectively, and the diagonal entries are given in (5.2d). One can check that with the above notations equation (2.2) holds and therefore we obtain a point $\mathfrak{m} \in \mathfrak{M}_d$ which depends on the free parameters $c_1, \ldots, c_d$. In particular, note that if $c_i \in (0, 1)$ for all $i \in \{0, 1, \ldots, d\}$ then $\tilde{c}_i \in (0, 1)$ for all $i \in \{1, \ldots, d\}$. Up to a permutation of the variables and the parameters, this choice leads to the multivariate Meixner polynomials defined in [12].

5.2. Fix now $q \notin \{0, 1\}$ and define

$$c_k = \tilde{c}_k = (1 - q)q^{k-1} \text{ for } k = 1, 2, \ldots, d. \qquad (5.3)$$

Then $c_0 = 1 - |c| = q^d$. We define also a $(d+1) \times (d+1)$ matrix $U = (u_{i,j})_{0 \le i,j \le d}$ with entries

$$u_{i,j} = 1, \qquad\qquad\qquad \text{when } i + j \le d, \qquad (5.4a)$$
$$u_{i,j} = \frac{1}{1 - q}, \qquad\qquad \text{when } i + j = d + 1, \qquad (5.4b)$$
$$u_{i,j} = 0, \qquad\qquad\qquad \text{when } i + j > d + 1. \qquad (5.4c)$$

With the above notations, we see that (2.2) holds leading to a point $\mathfrak{m} \in \mathfrak{M}_d$ which depends on the free parameter q. Moreover, if $q \in (0, 1)$ then $c_i = \tilde{c}_i \in (0, 1)$ for all

$i \in \{0, 1, \ldots, d\}$. These Meixner polynomials can be thought of as analogs of the multivariate Krawtchouk polynomials used in [1].

Acknowledgments

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